# Topic 12 -
# Power series solutions of ODEs

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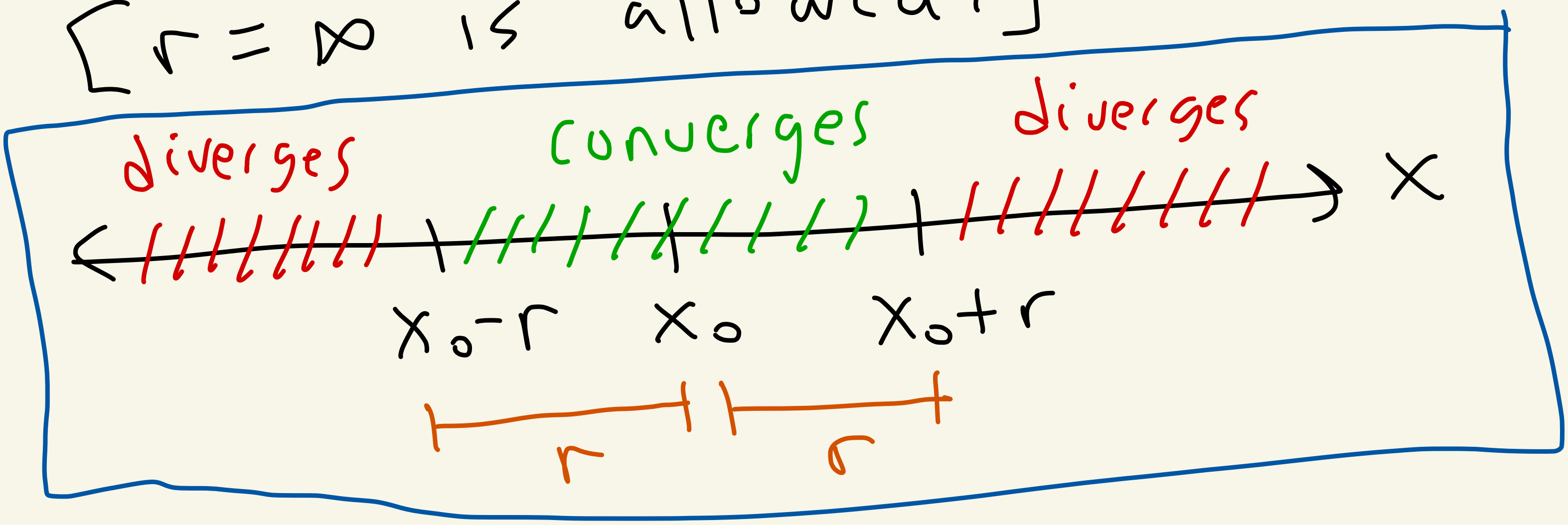
Def: We say that a function $f(x)$ is analytic at $x_0$ if it has a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

centered at $x_0$ with positive radius of convergence $r > 0$.

[$r = \infty$ is allowed.]



Ex: $x_0 = 0$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

has radius of convergence $r = \infty$

So, $\sin(x)$ is analytic at $x_0 = 0$

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Ex: $x_0 = 1$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

has radius of convergence $r = 1$.

Thus, $\frac{1}{x}$ is analytic at $x_0 = 1$.

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Ex: $x_0 = 2$

$$x^2 = 4 + 4(x-2) + (x-2)^2$$ last week

has radius of convergence $r = \infty$

Thus, $x^2$ is analytic at $x_0 = 2$

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**Facts:**

- polynomials are analytic for all $x_0$
- $e^x$, $\sin(x)$, $\cos(x)$ are analytic for all $x_0$
- rational functions (ratio of polynomials) are analytic at all $x_0$ except possibly where the denominator is zero.

Ex: $x^2 - 5x + 2$

is analytic for all $x_0$

It's a polynomial.

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Ex: $\dfrac{x}{x^2 - 1}$ ← rational function

is analytic for all $x_0$

except $\underbrace{x_0 = 1, -1}_{\text{when } x^2 - 1 = 0}$

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# Main Theorem

Consider either of the initial value problems:

$$\boxed{\begin{gathered} y' + a_0(x)y = b(x) \\ y(x_0) = y_0 \end{gathered}}$$

first order

OR

$$\boxed{\begin{gathered} y'' + a_1(x)y' + a_0(x)y = b(x) \\ y'(x_0) = y_0', \ y(x_0) = y_0 \end{gathered}}$$

second order

In either case, if the $a_i(x)$ and $b(x)$ are analytic at $x_0$ then there exists a unique solution to the initial-value problem of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

centered at $x_0$.

Furthermore, the radius of convergence $r > 0$ for the power series of the solution $y(x)$ is at least the smallest radius of convergence from amongst the power series of the $a_i(x)$ and $b(x)$.

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Ex: $$y' + \overbrace{a_1(x)}^{\substack{\text{radius of} \\ \text{convergence } r=2 \\ \text{at } x_0}} y = \overbrace{b(x)}^{\substack{\text{radius of} \\ \text{convergence} \\ r=3 \\ \text{at } x_0}}$$

$$y(x_0) = y_0$$

Then the solution will have radius of convergence at least $r = 2$.

Ex: Let's find a power series solution to

$$\boxed{\begin{aligned} &y' - 2xy = 0 \\ &y(0) = 1 \end{aligned}}$$

center at $x_0 = 0$

coefficients

$$-2x = \underbrace{0 - 2x + 0x^2 + 0x^3 + \cdots}_{\text{power series centered at } x_0 = 0}$$

$$0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$$

$r = \infty$

This tells us that we will have a power series solution

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

will radius of convergence $r = \infty$

We need to find $y^{(n)}(0)$ for $n \geq 0$.

Given:

$$\boxed{\begin{array}{c} y' - 2xy = 0 \\ y(0) = 1 \end{array}} \leftarrow y' = 2xy$$

So, $\boxed{y(0) = 1}$

And, $y'(0) = 2[0][y(0)] = 2[0][1] = 0$

So, $\boxed{y'(0) = 0}$

Differentiate $y' = 2xy$ with respect to $x$ to get:

$$y'' = 2y + 2xy'$$

$$\boxed{(fg)' = f'g + fg'}$$

So,

$$y''(0) = 2[y(0)] + 2(0)[y'(0)]$$
$$= 2[1] + 2(0)(0)$$
$$= 2.$$

So, $\boxed{y''(0) = 2}$

Differentiate $y'' = 2y + 2xy'$ to get

$$y''' = 2y' + 2y' + 2xy''$$
$$y''' = 4y' + 2xy''$$

So,

$$y'''(0) = 4[y'(0)] + 2(0)[y''(0)]$$
$$= 4(0) + 2(0)(2)$$
$$= 0$$

Thus, $\boxed{y'''(0) = 0}$

Differentiate $y''' = 4y' + 2xy''$ to get

$$y'''' = 4y'' + 2y'' + 2xy'''$$
$$= 6y'' + 2xy'''$$

So,

$$y''''(0) = 6[y''(0)] + 2(0)[y'''(0)]$$
$$= 6(2) + 2(0)(0)$$
$$= 12$$

Thus, $\boxed{y''''(0) = 12}$

So,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y''''(0)}{4!}x^4 + \dots$$

$$y(x) = 1 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 + \cdots$$

$$\boxed{\begin{array}{l} y(x) = 1 + x^2 + \frac{1}{2}x^4 + \cdots \\ \text{with radius of convergence } r = \infty \end{array}}$$

$\boxed{\text{Side note}}$ Using topic 3, you can show

$$y(x) = e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots$$

$\boxed{t = x^2}$

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

Ex: Consider

$$\boxed{\begin{aligned} &y'' + x^2 y' - (x-1)\,y = \ln(x) \\ &y'(1) = 0,\ y(1) = 0 \end{aligned}}$$

$x_0 = 1$

**Coefficients**

$$x^2 = 1 + 2(x-1) + (x-1)^2 + 0(x-1)^3 + \cdots$$

$$-(x-1) = 0 - 1\cdot(x-1) + 0(x-1)^2 + 0(x-1)^3 + \cdots$$

$r = \infty$

$$\ln(x) = (x-1) + \tfrac{1}{2}(x-1)^2 + \cdots$$

$r = 1$

So we can find a solution

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots$$

with radius of convergence is at least $r=1$.

We have

$$\boxed{\begin{aligned} y(1) &= 0 \\ y'(1) &= 0 \end{aligned}}$$

and

$$y'' = \ln(x) - x^2 y' + (x-1)y$$

$$y''(1) = \ln(1) - (1)^2[y'(1)] + (1-1)[y(1)]$$

$$= 0 - (1)[0] + (0)[0]$$

$$= 0$$

So,

$$y''(1) = 0$$

Differentiating above we get

$$y''' = \frac{1}{x} - 2xy' - x^2 y'' + (1)y + (x-1)y'$$

$$y'''(1) = \frac{1}{1} + 2(1)[y'(1)] - (1)^2[y''(1)] + y(1) + (1-1)[y'(0)]$$

$$= 1 + 2(1)[0] - (1)[0] + 0 + (0)(0)$$

$$= 1$$

Thus, $y'''(1) = 1$

One can calculate that

$$y''''(1) = -3$$

Thus,

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2$$

$$+ \frac{y'''(1)}{3!}(x-1)^3 + \frac{y'''(1)}{4!}(x-1)^4 + \cdots$$

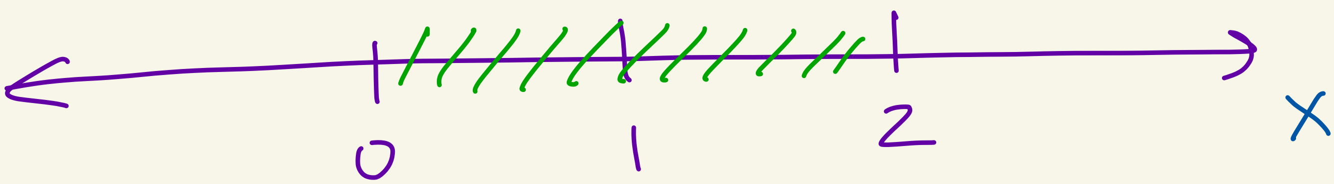
$$= 0 + 0(x-1) + 0(x-1)^2$$

$$+ \frac{1}{3!}(x-1)^3 - \frac{3}{4!}(x-1)^4 + \cdots$$

So,

$$\boxed{y(x) = \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^4 + \cdots}$$

with radius of convergence at least $r = 1$




converges at least here